Last Time: Orthogonality.

Gram-Schmidt Process:

Given lin. ind. vects. $v_1, v_2, \dots, v_k$ in $\mathbb{R}^n$, we can construct a set of mutually orthogonal vects $u_1, u_2, \dots, u_k$ with the same span. Formulaically:

$$\begin{cases} u_1 = v_1 \\ u_i := v_i - \mathrm{proj}_{\boxed{u_{i-1}}}(v_i) - \mathrm{proj}_{\boxed{u_{i-2}}}(v_i) - \cdots - \mathrm{proj}_{\boxed{u_1}}(v_i) \end{cases}$$

Ex: Apply GS-process to $v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $v_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$, $v_3 = \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix}$.

Sol: $u_1 = v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

$$u_2 = v_2 - \mathrm{proj}_{u_1}(v_2) = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} - \frac{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}}{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} - \frac{0}{3} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

$$u_3 = v_3 - \mathrm{proj}_{u_2}(v_3) - \mathrm{proj}_{u_1}(v_3)$$

$$= \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix} - \frac{\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix}}{\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} - \frac{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix}}{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix} - \frac{-1}{2} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} - \frac{4}{3} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 + \frac{1}{2} - \frac{4}{3} \\ 3 + 0 - \frac{4}{3} \\ 1 - \frac{1}{2} - \frac{4}{3} \end{pmatrix} = \begin{pmatrix} -5/6 \\ 5/3 \\ -5/6 \end{pmatrix} = \frac{5}{6} \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}$$

Check: $u_1 \cdot u_2 = 0$, $u_1 \cdot u_3 = 0$, $u_2 \cdot u_3 = 0$

$$u_1 \cdot u_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = 1 + 0 - 1 = 0$$

$$u_1 \cdot u_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \cdot \frac{5}{6} \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix} = \frac{5}{6}(-1 + 2 - 1) = \frac{5}{6} \cdot 0 = 0$$

$$u_2 \cdot u_3 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \cdot \frac{5}{6} \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix} = \frac{5}{6}(-1 + 0 + 1) = \frac{5}{6} \cdot 0 = 0 \qquad \square$$

Another check method: Note $u \cdot v = u^T v$

(i.e. $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} x & y & z \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = [xa + yb + zc]$)

Take $A = [u_1 | u_2 | u_3]$, check

$$A^T A = \begin{bmatrix} u_1^T \\ \hline u_2^T \\ \hline u_3^T \end{bmatrix} [u_1 | u_2 | u_3] = \begin{bmatrix} u_1^T u_1 & u_1^T u_2 & u_1^T u_3 \\ u_2^T u_1 & u_2^T u_2 & u_2^T u_3 \\ u_3^T u_1 & u_3^T u_2 & u_3^T u_3 \end{bmatrix}$$

$$= \begin{bmatrix} u_1 \cdot u_1 & u_1 \cdot u_2 & u_1 \cdot u_3 \\ u_2 \cdot u_1 & u_2 \cdot u_2 & u_2 \cdot u_3 \\ u_3 \cdot u_1 & u_3 \cdot u_2 & u_3 \cdot u_3 \end{bmatrix} = \begin{bmatrix} |u_1|^2 & 0 & 0 \\ 0 & |u_2|^2 & 0 \\ 0 & 0 & |u_3|^2 \end{bmatrix}$$

if the u_i's are mutually orthogonal...

Point: $A^T A$ is a diagonal matrix if columns of A are mutually orthogonal... Should do: normalize the columns of A (i.e. force $|u_i| = 1$ for all i by taking suitable scalar multiples), then we obtain an "orthogonal matrix".

Def^n: A matrix M is orthogonal when $M^T = M^{-1}$ (M is $n \times n$).

Prop: M is orthogonal if and only if the columns of M form an orthonormal basis for $\mathbb{R}^n$. pf: Easy exercise $\boxed{\checkmark}$.

Def^n: A basis of $\mathbb{R}^n$ is orthonormal when the elements are mutually orthogonal and all have length 1.

Ex: Moment ago: we computed $u_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $u_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$, $u_3 = \frac{5}{6}\begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}$ form an orthogonal basis of $\mathbb{R}^3$. However,

$|u_1| = \sqrt{1^2+1^2+1^2} = \sqrt{3}$, $|u_2| = \sqrt{1+1} = \sqrt{2}$,

$|u_3| = \frac{5}{6}\sqrt{1+4+1} = \frac{5}{6}\sqrt{6} = \frac{5}{\sqrt{6}}$, so $\mathcal{B} = \{u_1, u_2, u_3\}$ is not orthonormal. But

$$\hat{\mathcal{B}} = \left\{\frac{1}{|u_1|}u_1, \frac{1}{|u_2|}u_2, \frac{1}{|u_3|}u_3\right\} = \left\{\frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\-1\end{pmatrix}, \frac{\frac{5}{6}}{5/\sqrt{6}}\begin{pmatrix}-1\\2\\-1\end{pmatrix}\right\}$$

$$= \left\{\frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\-1\end{pmatrix}, \frac{1}{\sqrt{6}}\begin{pmatrix}-1\\2\\-1\end{pmatrix}\right\} \quad \underline{\underline{\text{IS}}} \text{ orthonormal} \ddot\smile$$

$$= \left\{\begin{pmatrix}1/\sqrt{3}\\1/\sqrt{3}\\1/\sqrt{3}\end{pmatrix}, \begin{pmatrix}1/\sqrt{2}\\0\\-1/\sqrt{2}\end{pmatrix}, \begin{pmatrix}-1/\sqrt{6}\\2/\sqrt{6}\\-1/\sqrt{6}\end{pmatrix}\right\}$$

<u>Check:</u>

$$\begin{bmatrix}1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6}\\ 1/\sqrt{3} & 0 & 2/\sqrt{6}\\ 1/\sqrt{3} & -1/\sqrt{2} & -1/\sqrt{6}\end{bmatrix}^T \begin{bmatrix}1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6}\\ 1/\sqrt{3} & 0 & 2/\sqrt{6}\\ 1/\sqrt{3} & -1/\sqrt{2} & -1/\sqrt{6}\end{bmatrix} \overset{???}{=} I_3$$

$$\begin{bmatrix}1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3}\\ 1/\sqrt{2} & 0 & -1/\sqrt{2}\\ -1/\sqrt{6} & 2/\sqrt{6} & -1/\sqrt{6}\end{bmatrix} \begin{bmatrix}1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6}\\ 1/\sqrt{3} & 0 & 2/\sqrt{6}\\ 1/\sqrt{3} & -1/\sqrt{2} & -1/\sqrt{6}\end{bmatrix}$$

$$= \begin{bmatrix}\frac{1}{3}+\frac{1}{3}+\frac{1}{3} & \frac{1}{\sqrt{2}\sqrt{3}}+0-\frac{1}{\sqrt{2}\sqrt{3}} & \boxed{-\frac{1}{\sqrt{3}\sqrt{6}}+\frac{2}{\sqrt{3}\cdot\sqrt{6}}-\frac{1}{\sqrt{3}\sqrt{6}}}\\ \frac{1}{\sqrt{2}\sqrt{3}}+0-\frac{1}{\sqrt{2}\sqrt{3}} & \frac{1}{2}+0+\frac{1}{2} & \frac{-1}{\sqrt{2}\sqrt{6}}+0+\frac{1}{\sqrt{2}\sqrt{6}}\\ \boxed{-\frac{1}{\sqrt{3}\sqrt{6}}+\frac{2}{\sqrt{3}\sqrt{6}}-\frac{1}{\sqrt{3}\sqrt{6}}} & \frac{1}{\sqrt{2}\sqrt{6}}+0+\frac{1}{\sqrt{2}\sqrt{6}} & \frac{1}{6}+\frac{4}{6}+\frac{1}{6}\end{bmatrix}$$

$$= \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} \quad \checkmark \quad \ddot\smile$$

$\boxed{//}$

<u>Remark</u>: This will always compute an orthonormal basis from orthogonal one.

<u>Algorithm (Extended Gram-Schmidt Process)</u>: Given $v_1, v_2, \ldots, v_k$ lin. indep. in $\mathbb{R}^n$. To compute an orthonormal collection w/ same span:

① Apply the Gram-Schmidt Process to $v_1, v_2, \ldots, v_k$.

② Normalize each output vector (i.e. scale each u_i by $\frac{1}{|u_i|}$).

Ex: Apply Extended GS process to $v_1 = \begin{pmatrix}1\\0\\-1\end{pmatrix}$, $v_2 = \begin{pmatrix}0\\3\\1\end{pmatrix}$, $v_3 = \begin{pmatrix}1\\1\\1\end{pmatrix}$.

(NB: Compare w/ previous example to note order matters for GS-process!)

Sol: $u_1 = v_1 = \begin{pmatrix}1\\0\\-1\end{pmatrix}$

$$u_2 = \begin{pmatrix}0\\3\\1\end{pmatrix} - \mathrm{proj}_{\begin{pmatrix}1\\0\\-1\end{pmatrix}}\begin{pmatrix}0\\3\\1\end{pmatrix} = \begin{pmatrix}0\\3\\1\end{pmatrix} - \frac{\begin{pmatrix}1\\0\\-1\end{pmatrix}\cdot\begin{pmatrix}0\\3\\1\end{pmatrix}}{\begin{pmatrix}1\\0\\-1\end{pmatrix}\cdot\begin{pmatrix}1\\0\\-1\end{pmatrix}}\begin{pmatrix}1\\0\\-1\end{pmatrix} = \begin{pmatrix}0\\3\\1\end{pmatrix} - \frac{-1}{2}\begin{pmatrix}1\\0\\-1\end{pmatrix}$$

$$= \begin{pmatrix}0+\frac{1}{2}\\3+0\\1-\frac{1}{2}\end{pmatrix} = \begin{pmatrix}1/2\\3\\1/2\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1\\6\\1\end{pmatrix}$$

$$u_3 = \begin{pmatrix}1\\1\\1\end{pmatrix} - \mathrm{proj}_{\frac{1}{2}\begin{pmatrix}1\\6\\1\end{pmatrix}}\begin{pmatrix}1\\1\\1\end{pmatrix} - \mathrm{proj}_{\begin{pmatrix}1\\0\\-1\end{pmatrix}}\begin{pmatrix}1\\1\\1\end{pmatrix}$$

$$= \begin{pmatrix}1\\1\\1\end{pmatrix} - \frac{\cancel{\tfrac{1}{2}}\begin{pmatrix}1\\6\\1\end{pmatrix}\cdot\begin{pmatrix}1\\1\\1\end{pmatrix}}{\cancel{\tfrac{1}{2}}\begin{pmatrix}1\\6\\1\end{pmatrix}\cdot\cancel{\tfrac{1}{2}}\begin{pmatrix}1\\6\\1\end{pmatrix}}\cancel{\tfrac{1}{2}}\begin{pmatrix}1\\6\\1\end{pmatrix} - \frac{\begin{pmatrix}1\\0\\-1\end{pmatrix}\cdot\begin{pmatrix}1\\1\\1\end{pmatrix}}{\begin{pmatrix}1\\0\\-1\end{pmatrix}\cdot\begin{pmatrix}1\\0\\-1\end{pmatrix}}\begin{pmatrix}1\\0\\-1\end{pmatrix}$$

$$= \begin{pmatrix}1\\1\\1\end{pmatrix} - \frac{8}{38}\begin{pmatrix}1\\6\\1\end{pmatrix} - \frac{0}{2}\begin{pmatrix}1\\0\\-1\end{pmatrix}$$

$$= \begin{pmatrix}1\\1\\1\end{pmatrix} - \frac{4}{19}\begin{pmatrix}1\\6\\1\end{pmatrix} = \frac{1}{19}\begin{pmatrix}19-4\\19-24\\19-4\end{pmatrix} = \frac{1}{19}\begin{pmatrix}15\\-5\\15\end{pmatrix} = \frac{5}{19}\begin{pmatrix}3\\-1\\3\end{pmatrix}$$

$\mathrm{Proj}_{cu}(v) = \mathrm{Proj}_{u}(v)$

Span is equal!

GS Process yields $B = \left\{\begin{pmatrix}1\\0\\-1\end{pmatrix}, \boxed{\tfrac{1}{2}}\begin{pmatrix}1\\6\\1\end{pmatrix}, \boxed{\tfrac{5}{19}}\begin{pmatrix}3\\-1\\3\end{pmatrix}\right\}$. Normalizing,

$$\hat{B} = \left\{\frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\-1\end{pmatrix}, \frac{1}{\sqrt{38}}\begin{pmatrix}1\\6\\1\end{pmatrix}, \frac{1}{\sqrt{19}}\begin{pmatrix}3\\-1\\3\end{pmatrix}\right\}$$

$|c\vec{u}| = |c||\vec{u}|$ so normalizing $c\vec{u} \rightsquigarrow \frac{1}{|c\vec{u}|}c\vec{u} = \frac{c}{|c||\vec{u}|}\vec{u} = \frac{1}{|\vec{u}|}\vec{u}$ ↳ when $c > 0$

Q: Why are we computing orthonormal bases?

A: Orthonormal collections give vectors very nice representations...

In the GS process: $u_i = v_i - \sum \mathrm{proj}_{u_j}(v_i)$

$\rightsquigarrow v_i = \sum c_j u_j$

when u_j's orthonormal

So constant $c_j u_j = \mathrm{proj}_{u_j}(v_i) = \frac{u_j \cdot v_i}{u_j \cdot u_j} u_j = (u_j \cdot v_i) u_j$

Ex: The standard basis is an orthonormal basis.

$$v = \begin{pmatrix} a \\ b \\ c \end{pmatrix} = a e_1 + b e_2 + c e_3 = (v \cdot e_1) e_1 + (v \cdot e_2) e_2 + (v \cdot e_3) e_3$$

Point: orthonormal bases generalize the standard basis :)

Ex: Compute $\mathrm{Rep}_{\hat{B}}\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$ where $\hat{B} = \left\{ \underbrace{\frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}_{u_1}, \underbrace{\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}}_{u_2}, \underbrace{\frac{1}{\sqrt{6}}\begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}}_{u_3} \right\}$.

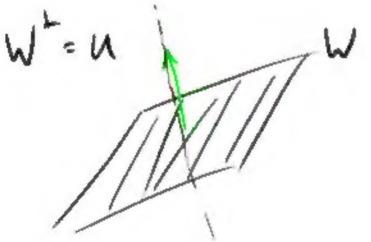
Sol: $v = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$ has

$$\left.\begin{aligned} u_1 \cdot v &= \tfrac{1}{\sqrt{3}}(2+1+2) = \tfrac{5}{\sqrt{3}} \\ u_2 \cdot v &= \tfrac{1}{\sqrt{2}}(2+0-2) = 0 \\ u_3 \cdot v &= \tfrac{1}{\sqrt{6}}(-2+2-2) = -\tfrac{2}{\sqrt{6}} \end{aligned}\right\} \rightsquigarrow \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} = \frac{5}{\sqrt{3}} u_1 + 0 u_2 - \frac{2}{\sqrt{6}} u_3.$$

ORTHOGONAL COMPLEMENTATION

Defn: A *complement* of subspace $W \leq V$ is a subspace U such that every vector of V can be expressed uniquely as $v = w + u$ where $w \in W$ and $u \in U$.

Picture:



$\leq \mathbb{R}^3$

Prop: If $W \leq \mathbb{R}^n$, then $W^\perp = \{ u \in \mathbb{R}^n : u \cdot w = 0 \text{ for all } w \in W \}$ is the complement of W.

Proof: Every basis of W extends to a basis of $\mathbb{R}^n$. Pick B a basis of W. Apply Extended GS to obtain $\hat{B}$. $\hat{B}$ is still a basis of W. Extend to $A = \hat{B} \cup D$ a basis for $\mathbb{R}^n$. $\hat{A} = \hat{B} \cup \hat{D}$. $W = \mathrm{span}(\hat{B})$ and $W^\perp = \mathrm{span}(\hat{D})$. $\square$

Computationally: to compute $W^{\perp}$:

① express $W = \mathrm{Col}(A)$ for matrix A.

② $W^{\perp} = \mathrm{null}(A^{T})$

Point → use A = matrix of any basis :) $\boxed{//}$